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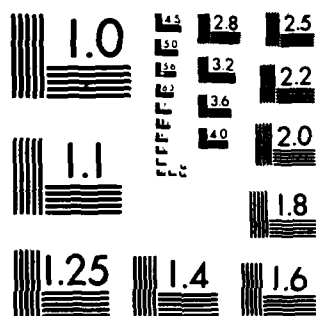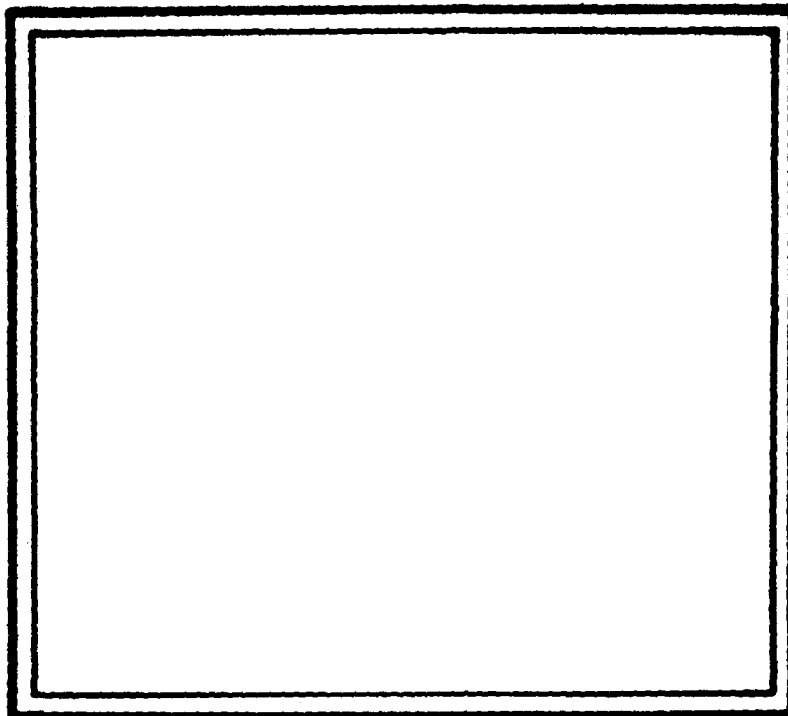UNCLASSIFIED   F/G 12/1   NL

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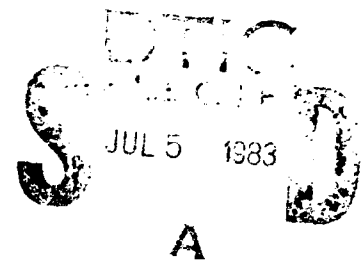## Newark, Delaware

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ASYMPTOTIC METHODS IN
RELIABILITY THEORY:

A REVIEW

by

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A considerable part of this review is based on the sources which were originally published in Russian and are available in the English translation.

ASYMPTOTIC METHODS IN RELIABILITY THEORY: A REVIEW.

Abstract

Section 1 of this paper reviews some works related to reliability evaluation of nonrenewable systems. The assumption that element failure rates are low allows to obtain an expression for the main term in the asymptotic representation of system reliability function. Section 2 is devoted to renewable systems. The main index of interest in reliability is the time to the first system failure. A typical situation in reliability is that the repair time is much smaller than the element lifetime. This "fast repair" property leads to an interesting phenomenon that for many renewable systems the time to system failure converges in probablity, under appropriate norming, to exponential distribution .Some basic theorems explaining this fact are presented and a series of typical examples is considered. Special attention is paid to reviewing the works describing the exponentiality phenomenon in the birth-and-death processes. Some important aspects of computing the normalizing constants are considered, among them, the role played by so-called "main event". Section 2 contains also a review on various bounds on the deviation from exponentiality. Sections 3 , 4 describe some additional aspects of asymptotics in reliability. It is typical for the probabilistic models considered in these sections, that a small parameter is introduced in an explicit form into the characteristic of the random processes.

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Key words: Reliability; exponentiality; fast repair; small parameter; main event; renewable systems; asymptotics.

0. Introduction

The ultimate goal of reliability theory is to give a numerical estimate of reliability indices. It is well known that for all more or less complicated cases an exact reliability evaluation is practically impossible. This stimulates interest to approximate methods in reliability calculations.

In many reliability models of great practical interest "small parameters" usually are present, e.g., a system under investigation has low element failure rates and/or the element failure rates are much smaller than their repair rate. This circumstance makes it possible to use efficient and powerful asymptotic methods for reliability computation.

The goal of this paper is to review a collection of works devoted to asymptotic reliability analysis. Most of these works were published in English translations of Russian scientific journals and for some reason are not very familiar to the Western applied probability community.

The contents of this paper follows.

A short Section 1 is devoted to an asymptotic analysis of coherent systems without renewal. Reliability analysis for these systems uses the fact that element failure rates, λ_i, are small in highly reliable systems (formally, $\lambda_i = \bar{\lambda}_i \theta$, $\theta \to 0$), and this makes it possible to compute the main term of the asymptotic expression for $R(t)$, the probability of failure free operation during time t.

Section 2 which is the central in this paper deals with renewable systems. The reliability index of greatest interest for these systems is

the probability of failure-free operation during a given mission time t. The asymptotic analysis exploits the basic fact that the average repair time for a failed element is usually many times smaller than the average failure-free operation time for the same element. (This is termed in reliability as "fast" or "rapid" repair). A very general scheme in asymptotical analysis is the following.

System functioning is described by a regenerating random process $\kappa(t)$, $\kappa(t)$ often being the number of failed elements at time t. The regeneration period of $\kappa(t)$, $\xi = \xi' + \xi''$, where ξ' corresponds to $\kappa(t) = 0$ and ξ'' to $\kappa(t) > 0$. Rapid repair results in a small probability q of having system failure on a single ξ''-interval. The time τ to system failure (SF), $\tau = \xi_1 + \ldots + \xi_N$, where N is a geometrically distributed random variable (generally, N depends on ξ). It is not surprising that by means of an appropriate norming constant γ, $\gamma \cdot \tau$ should converge in distribution to an exponential distribution function. Sections 2.1, 2.2 and 2.3 consider various aspects of this fact and related facts. Section 2.4 gives a review of a series of important applications to various reliability problems. Computation of the quantities determining the normalizing factor γ is a difficult analytic problem. We review briefly several important works on this topic, mainly due A.D. Solovyev, to give an idea about some technical aspects of these computations.

Many of the reliability models of renewable systems can be formulated in terms of queueing theory. The difference is that the asymptotic

analysis in reliability deals with low traffic, while the classical
queueing asymptotic deals with high traffic. A few works about low
traffic for queueing models will be reviewed in section 2.5.

Birth-and-Death processes are perhaps the most popular and useful
in reliability theory. Section 2.6 is devoted to a brief review of
the main features of the asymptotic analysis for these processes. Here
the failure-free operation time is interpreted as a passage time τ_{0m}
from an initial state 0 to a "high"-level state m representing
SF. The works by A.D. Solovyev and J. Keilson contain a rather complete
investigation of asymptotic properties of τ_{0m} and related variables.

Section 2.7 describes some general features of exponential approximation.
One of them is that the main term in the asymptotic expansion of the
normalizing factor has a transparent and simple probabilistic meaning
and corresponds to so-called "main event". For example, it might be
the event that no repair of any failed element was completed before SF
took place.

Section 2.8 surveys several works on estimating the error bounds
for the exponential approximation.

Section 3 gives very briefly an idea about a method developed by
I.N. Kovalenko for a special analysis of a multidimensional Markov-type
process which has one slowly varying component and one rapidly varying
component. It turns out that, under certain assumptions, the slowly
varying component behaves like a continuos time

Markov process. Section 4 describes a method of asymptotic analysis

based on a combination of analytical and simulation methods, which is applicable when the quantities determining the original process are expressed in a series form involving a small parameter.

We did not survey a large number of works devoted to the investigation of reliability indices for large t $(t \to \infty)$, e.g., stationary probabilities of being in a certain state, etc. In addition, all works dealing with asymptotic properties of input flows, e.g., flow thinning, superposition of flows, etc., were left outside the scope of this review.

Our goal was not to present an exhaustive list of bibliography on the topics concerned in this survey. First, we avoided mentioning those works which are not available in English. Second, we did not mention papers or books which already were cited in the reviewed sources. The reader can easily trace them from the cited works, if he becomes interested in the topics discussed in this review.

This review is addressed mainly to those readers who prefer to get a first acquaintance with a new topic on an intuitive level, without going into too may technical details. The reader who is interested in proofs and other formal details will be able to learn them from the reviewed sources.

Notation and Abbreviations

We shall use the following notation and abbreviations.

r.v.   -   random variable

p.d.f. -   probability density function

d.f.   -   (cumulative) distribution function

BD-process - Birth-and-Death process

SMP   -   Semi-Markov process

SF   -   system failure

CM   -   complete monotone (family of distributions)

$\xi \sim F(x)$ - r.v. ξ is distributed according to the d.f. $F(x)$.

$\xi \sim \text{Exp}(\lambda)$ - r.v. ξ has an exponential distribution with parameter λ.

$a \sim b$ - a and b are asymptotically equal, i.e., $a/b \to 1$

$E[\zeta]$ -   expectation of r.v. ζ.

$\bar{F}(x) = 1 - F(x)$

$\underline{x} = (x_1, x_2, \ldots, x_n)$ - a row vector; $\underline{x} \leq \underline{y} \iff x_i \leq y_i$, $i=1,\ldots,n$, but for some $j, x_j < y_j$.

τ -   time to system failure.

$R(t) = P\{\tau > t\}$ - system reliability.

λ, λ_i -   failure rates and/or transition rates for a Markov process.

E_+, E_- - sets of "good", resp. "bad" states of a system.

$\nu(t), \kappa(t), \underline{x}(t)$ - random processes.

$P = ||p_{ij}||$ - transition matrix for a Markov chain.

π_i - stationary probabilities for a Markov chain.

$F(x), H(x), G(x)$ - d.f.'s.

τ_{0m} - passage time from state 0 to state m in a random process.

$\bar{\tau}_{0m}$ - the expectation of τ_{0m}

\# - end of proof or end of the formulation of a theorem.

$|B|$ - the number of elements in the set B.

$\underline{1} = (1,1,\ldots,1)'$ - a unit column vector.

1. Systems Without Renewal

Consider an arbitrary coherent system consisting of n independent components (elements), the lifetime of ith element being an exponential r.v. with failure rate λ_i. Let the state of the system be described by a binary function $\varphi(\underline{x})$, where $\underline{x} = (x_1, x_2, \ldots, x_n)$, x_i is a binary indicator of the state of ith element (see Barlow & Proschan, 1975, Ch. 1,2,4). Let τ be the system lifetime and let t be the mission time. Denote for any $\underline{x}$, $A(\underline{x}) = \{j : x_j = 1\}$, $B(\underline{x}) = \{j : x_j = 0\}$. Then

$$P\{\tau \leqslant t\} = F(t) = \sum_{\{\underline{x} : \varphi(\underline{x}) = 0\}} \prod_{j \in A(\underline{x})} e^{-\lambda_j t} \prod_{j \in B(\underline{x})} (1 - e^{-\lambda_j t}) . \qquad (1.$$

An asymptotic investigation of this formula for low failure rates was done by Burtin & Pittel, 1972. A small parameter θ is introduced by representing $\lambda_i = \bar{\lambda}_i \theta$ and setting $\theta \to 0$. Denote by $\Lambda(t|\theta)$ system failure rate, $\Lambda(t|\theta) = (dF(t)/dt)/R(t)$ and by r the size of minimal cut-set: $r = \min\{|B(\underline{x})| : \varphi(\underline{x}) = 0\}$. The following theorem was proved in the abovecited paper.

Theorem (Burtin & Pittel, 1972). As $\theta \to 0$

$$\Lambda(t|\theta) = r\theta^r t^{r-1} \sum_{\{\underline{x} : \varphi(\underline{x}) = 0 \ \& \ |B(\underline{x})| = r\}} \prod_{j \in B(\underline{x})} \bar{\lambda}_j (1 + o(1)) \qquad (1.2)$$

uniformly with respect to an arbitrary interval $0 < \delta \leqslant t < \Delta < \infty$. #

The main term in (1.2) corresponds to the Weibull distribution. To explain this surprising fact, let us substitute $\lambda_j = \bar{\lambda}_j \theta$ into (1.1) and expand $\exp(-\lambda_j t) = 1 - \bar{\lambda}_j \theta t + o(\theta)$. The main term in the

expansion will be determined by failure states $\underline{x}$ such that $\varphi(\underline{x}) = 0$ and $|B(\underline{x})| = r$.

After some algebra one obtains from (1.1) that

$$P\{\tau \leq t\} = 1 - t^r \theta^r g(\underline{\bar{\lambda}}) \, (1 + o(1)) \, , \tag{1.3}$$

where $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and

$$g(\underline{\bar{\lambda}}) = \sum_{\{\underline{x}: \varphi(\underline{x}) = 0 \ \& \ |B(\underline{x})| = r\}} \prod_{j \in B(\underline{x})} \bar{\lambda}_j \, . \tag{1.4}$$

Thus,

$$R(t) \sim 1 - t^r \theta^r g(\underline{\bar{\lambda}}) \sim \exp\{-\theta^r t^r g(\underline{\bar{\lambda}})\} \, , \tag{1.5}$$

which corresponds to the Weibull distribution.

The role of min-cuts in getting an accurate approximation to system reliability was discussed by Locks, 1980.

The formula (1.2) and/or (1.4, 1.5) can be used for practical purposes only if there is an efficient method for finding all minimal-size cut-sets. For a real-size system this might be a rather difficult problem.

Z. Waksman, 1982, proposed a promising approach to this problem, for a particular type of coherent structures, namely, two-terminal networks. His approach uses a flow-in-network technique and is based on introducing a special subnetwork which is equivalent to the original one in the sense of its asymptotic analysis.

Consider a network - an undirected multigraph- $G = (E,V)$ - for which the failure is defined as disconnection between two vertices S and T, called terminals. A separator $\tilde{E}$ in G is a set of edges $\tilde{E} \subseteq E$ such that the removal of $\tilde{E}$ disconnects all chains joining S with T. Separators of minimal size are called minimum separators (this corresponds to the minimal cut sets in Barlow & Proschan terminology). An edge $e \in \tilde{E}$ is called essential iff e is contained in a minimum separator. Z. Waksman introduces a subnetwork $G^+ = (E^+,V^+)$ obtained from G by <u>contracting</u> all edges of G which are not essential. It turns out that the main terms in asymptotic failure probability representation (1.3) are the same for G and G^+.

Assume that each $e \in E$ has a maximal flow capacity equal to one. It was proved that there is a maximum flow from S to T which saturates each essential edge and does not saturate any unessential edge. The above-cited paper contains an efficient flow-type algorithm for identifying all essential edges. The reduction of G to G^+ already signifies a considerable simplification in reliability computation. The evaluation of the main term in the asymptotic expansion for G^+ is carried out by an efficient algorithm which uses pivoting decomposition (see Barlow & Proschan, 1975, Ch. 1). Specific properties of G^+ allow accelerate considerably the computations.

It is easy to obtain simple formulas for $g(\underline{\lambda})$ for several important particular cases. For example, for a k-out-of-n system,

$$g(\underline{\lambda}) = \sum_{\{i_1,\ldots,i_r\}} \prod_{k=1}^{n} \lambda_{i_k} \quad , \qquad (1.6)$$

where $r = n-k+1$ and the summation is made over all $\binom{n}{r}$ combinations of r indices. Derivation of (1.6) and also formulas for series-parallel structures can be found, for example, in Gertsbakh, 1982.

The formula (1.5) for $R(t)$ allows obtaining a lower confidence limit for $R(t)$ if an upper confidence limit (UCL) can be found for the parametric function $g(\underline{\lambda})$. A method of constructing such an UCL was described in Gertsbakh, 1982.

It is important for practical applications to estimate the error in reliability evaluation by formula (1.5). In some situations, the following upper bound might provide good results:

$$0 \leqslant F(t) - \left\{ \sum_{\{x:\varphi(\underline{x})=0 \ \& \ |B(\underline{x})|=r\}} \prod_{j \in A(x)} e^{-\lambda_j t} \prod_{j \in B(x)} (1-e^{-\lambda_j t}) \right\} \leqslant$$

$$\leqslant e^{-\Lambda} \sum_{m=r+1}^{\infty} \frac{\Lambda^m}{m!} \tag{1.7}$$

(here $\Lambda = t \sum_{i=1}^{n} \lambda_i$) .

This formula can be obtained by combining the following two facts. First, the markovian process $\nu(t)$ which describes system degradation from state $(1,1,\ldots,1)$ down to a failure state $\underline{x}^* \in E_-^* = \{\underline{x}:\varphi(\underline{x})=0 \ \& \ |B(\underline{x})| \geqslant r+1\}$ should have at least $(r+1)$ transitions. Second, the probability of having at least $(r+1)$ transitions in $\nu(t)$ is no less than the same probability in a "majorant" process $\tilde{\nu}(t)$ which has a constant transition rate $\tilde{\Lambda} = \sum_{i=1}^{n} \lambda_i$. But the latter is a Poisson

process and that explains the form of the bound in (1.7). This method, in fact, was suggested in a general form by Kovalenko, 1975, Sections 2.6, 2.7.

2. Systems with Renewal

2.1. <u>Introduction</u>. Let us consier a system consisting of two identical blocks 1 and 2, having exponential lifetime, and one repair channel. The repair time for both blocks has some d.f. $H(x)$. At $t = 0$ block 1 starts working while block 2 is in standby. When block 1 fails, block 2 starts working and block 1 goes to repair. If block 2 fails during the repair, the system, by definition, had failed. Otherwise, block 1 returns to operation and block 2 - to standby. We are interested in the distribution of the time to failure τ of the system. Let us call a cycle the time interval which starts by putting block 1 into operation and ends with the completion of repair of a failed block (type 1 cycle), or with SF (type 2 cycle). Denote by ξ, η the lengths of these cycles. It is clear that

$$\tau = \xi_1 + \xi_2 + \ldots + \xi_{N-1} + \eta , \qquad (2.1)$$

where N is the random number of cycles, ξ_i are independent realizations of r.v. ξ. Let q be the probability that the cycle will terminate by a SF. Then clearly N has a geometric distribution:

$$P\{N=k\} = (1-q)^{k-1}q, \qquad k \geq 1 . \qquad (2.2)$$

The operation of many renewable systems until the appearance of the first SF can be described in terms of the model (2.1). It is very

important for reliability theory to know the probabilistic properties of the r.v. τ. It is typical for situations studied in reliability that q is small, which reflects the fact that the average time of normal operation is much larger than the average renewal (repair) time. Therefore, it is natural to investigate the distribution of τ for small values of q, formally for $q \to 0$. It turns out that under quite general conditions the appropriately normalized r.v. τ converges in distribution to an exponential d.f. with parameter $\lambda = 1$.

2.2. <u>Keilson's theorem</u>. Let $\nu(t)$ be an ergodic continuous time regenerating random process with state space X. A particular state $x_0 \in X$ has the property that each entrance into it is a regeneration point of $\nu(t)$. Define for each natural m a decomposition $X = X_1^m + X_2^m$. Suppose $x_0 \in X_1^m$. Let $(1-q_m)$ be the probability that there will be no SF on a single regeneration period, where SF is defined as entering the set X_2^m. Denote by μ, $\mu < \infty$, the expectation of the length of one regeneration cycle. Keilson, 1966, proved the following theorem.

<u>Theorem (Keilson, 1966, 1979)</u>. Let $q_m \to 0$ as $m \to \infty$ and let $\tau_m = \inf\{t: \nu(t) \in X_2^m | \nu(0) = x_0\}$. Then for each $x > 0$

$$\lim_{m \to \infty} P\{\frac{q_m \tau_m}{\mu} \le x\} = 1 - e^{-x} \qquad \# \qquad (2.3)$$

The proof is based on the investigation of the limiting form of the moment generating function for r.v. τ_m. The importance of this theorem

stems from the fact that only two parameters are involved in the limiting distribution: the average length of a regeneration period μ and "failure" probability q_m on a single regeneration period.

Remark. It follows from the proof of the theorem that μ in (2.3) can be replaced by μ_m, the average (conditional) return time to x_0 for those trajectories which do not visit X_2^m.

We demonstrate an application of Keilson's theorem by an example which might be of interest for reliability theory.

Example. Asymptotic distribution of the time until a SMP gets out of a kernel (Ushakov & Pavlov, 1978).

Consider a SMP $\nu(t)$ with state space $X = (1,2,3,\ldots)$. Let $F_{ij}(t)$ be the one-step sojourn d.f.'s and let $P = ||p_{ij}||$ be the matrix of transition probabilities. Let $X = X_1 + X_2$, $X_1 = \{1,2,\ldots,n\}$. X_1 is called a "kernel". Define for each $i \in X_1$, $\epsilon_i = \sum_{j \in X_2} p_{ij}$. It is assumed that ϵ_i are small and let formally $\epsilon_i \to 0$. The Markov chain with state space X and transition matrix P has no transient states, is irreducible and positive recurrent. Assume also that for $i \in X$

$$m_i = \sum_{j \in X} p_{ij} \int_0^\infty \bar{F}_{ij}(t)dt \leq \text{Const} < \infty \qquad (2.4)$$

Let us introduce a "conditional" SMP $\nu_0(t)$ with state space X_1. $\nu_0(t)$ is obtained from $\nu(t)$ by setting $\epsilon_i = 0$ and $p_{ij}^0 = p_{ij}/(1-\epsilon_i)$. Assume that $\nu_0(t)$ is an ergodic process. Denote by π_j^0 the stationary probabilities for the Markov chain $P^0 = ||p_{ij}^0||$

By a well known formula the mean return time μ_{11}^0 to the state 1 in $\nu_0(t)$ is $\mu_{11}^0 = \sum_{i \in X_1} \pi_i^0 m_i^0 / \pi_1^0$, where $m_i^0 = \sum_{j \in X_1} p_{ij}^0 \int_0^\infty \overline{F}_{ij}(t)dt$.

We are interested in the limiting distribution of r.v.

$\tau = \inf\{t : \nu(t) \in X_2 | \nu(0) = 1\}$ which is the exit time of $\nu(t)$ from the kernel. The key observation is that τ has the representation (2.1), where ξ_i are the return times to state 1 without visiting X_2 and η is a "direct" passage time from state 1 to X_2.

There is a small probability q that SMP $\nu(t)$ will leave the kernel between two consecutive visits into state 1, because, as postulated, ϵ_i are small. Thus, Keilson's theorem can be applied and it remains only to compute q. To do that, let us introduce the following conditional probabilities:

$\alpha_i = P\{$exit to X_2 during a single regeneration period, before returning to state 1 $|$ the regeneration period starts at state $i \in X_1\}$.

Clearly, α_i satisfy the following system:

$$\alpha_i = \epsilon_i + \sum_{k=2}^n p_{ik} \alpha_k, \quad i = 1,\ldots,n . \tag{2.5}$$

Dividing each equation by $1-\epsilon_i$, we obtain

$$\alpha_i + \alpha_i \epsilon_i + O(\epsilon_i^2) = \sum_{k=1}^n p_{ik}^{(0)} \alpha_k - p_{i1}^{(0)} \alpha_1 + \epsilon_i . \tag{2.6}$$

Now multiply i^{th} equation by π_i^0 and sum up all equations. Using the fact that $\underline{\pi}^0 = \underline{\pi}^0 P^0$, $\underline{\pi}^0 \underline{1} = 1$, one obtains

15

$$\sum_{i \in X_1} \pi_i^0 \alpha_i + \sum_{i \in X_1} \pi_i^0 \alpha_i \epsilon_i + O(\epsilon^2) = \sum_{i \in X_1} \pi_i^0 \alpha_i - \pi_1^0 \alpha_1 + \sum_{i \in X_1} \pi_i^0 \epsilon_i ,$$

where $\epsilon = \max_{i \in X_1} \epsilon_i$. From this

$$\alpha_1 = (\sum_{i \in X_1} \pi_i^0 \epsilon_i - \sum_{i \in X_1} \pi_i^0 \alpha_i \epsilon_i + O(\epsilon^2))/\pi_1^0 . \tag{2.7}$$

But from the system (2.5) one can see that $\alpha_i = O(\epsilon_i)$. Using (2.7), we see that

$$\alpha_1 = q = \frac{\sum_{i \in X_1} \pi_i^0 \epsilon_i}{\pi_i^0} + O(\epsilon^2) \sim \frac{\sum_{i \in X_1} \pi_i^0 \epsilon_i}{\pi_1^0} . \tag{2.8}$$

Thus, the normalizing factor in (2.3) can be taken as

$$\beta = \frac{q}{\mu_{11}} = \sum_{i \in X_1} \pi_i^0 \epsilon_i / \sum_{i \in X_1} \pi_i^0 m_i^0 , \tag{2.9}$$

and we obtain finally that

$$\lim P\{\beta \tau \leq x\} = 1 - e^{-x} \tag{2.10}$$

when $\epsilon_i \to 0$ for $i \in X_1$.

This result was obtained by Pavlov&Ushakov , 1978, using more complicated techniques.

2.3. Generalizations of Keilson's model. Solovyev's theorems.

Keilson's theorem relates the fact that $q_m \to 0$ as $m \to \infty$ to the changes in X_2^m. One can think, for example, that X_2^m get "smaller" when $m \to \infty$. But this particular form of the behavior of q_m is too restrictive. It is more natural to consider a regenerating processes that can change in such a way that the probability of appearance of the SF on a single regeneration period goes to zero. This approach was adopted by A.D. Solovyev.

Let $\kappa(t)$ be a regenerating process, and $t_0 = 0 < t_1 < t_2 < \ldots < t_n < \ldots$, instants of regenerations. On each regeneration period $\xi_n = t_n - t_{n-1}$ an event A_n can occur at some instant $t_{n-1} + \eta_n$, $0 < \eta_n \leq \xi_n$. A_n and η_n are defined on the trajectories $\{\kappa(t),\ t_{n-1} < t \leq t_n\}$ and they are independent of the behavior of $\kappa(t)$ outside this period. Let r.v. τ be the time of the first appearance of event A and let χ_n be the indicator function of A_n. Define

$$\zeta_n = \begin{cases} \xi_n , & \text{if } \chi_n = 0 ; \\[2mm] \eta_n , & \text{if } \chi_n = 1 . \end{cases} \tag{2.11}$$

Let $\zeta_n \sim F(x)$, $\varphi_-(z) = E[\exp(-z\zeta_n)\chi_n]$, $\varphi(z) = E[\exp(-z\zeta_n)]$, $q = \varphi_-(0) = P\{A_n\}$. It is easy to obtain that

$$a(z) = E[e^{-z\tau}] = \varphi_-(z)/(1 + \varphi_-(z) - \varphi(z))$$

Let $\bar{q} = \sup_{z>0} \dfrac{q-\varphi_-(z)}{1-\varphi(z)}$, $q_0 = \max(q,\bar{q})$.

For the sake of simplicity, the subscripts of r.v.'s ζ_n, ξ_n, η_n, χ_n will be dropped. Denote also

$$\alpha_p = \left(\frac{E[\zeta^p]}{(E[\zeta])^p}\right)^{1/(p-1)} \cdot q, \quad \bar{\alpha}_p = \left(\frac{E[\xi^p]}{(E[\xi])^p}\right)^{1/(p-1)} \cdot q . \tag{2.12}$$

Theorem 1 (Solovyev, 1971)

If the distribution of (ξ,η,χ) vary in such a way that $q > 0$ and $q_0 \to 0$ and if for some normalizing factor γ, $\gamma\tau$ converges to a proper r.v., then

$$\lim E[e^{-\gamma z\tau}] = (1 + \omega(z))^{-1} , \tag{2.13}$$

where

$$\omega(z) = \int_0^\infty \frac{1-e^{-zx}}{x} \, dP(x) , \quad P(x) \text{ is nondecreasing function such that}$$

$P(0) = 0$ and $\int_1^\infty \dfrac{dP(x)}{x} < \infty$. A necessary and sufficient condition for the distribution (2.13) to converge is that for every $x > 0$,

$$\lim \int_0^x \frac{t}{q} \, dF\left(\frac{t}{\gamma}\right) = P(x) \qquad \#$$

Remark (Solovyev, 1971). In the case $E[\zeta] = T < \infty$ the normalizing factor can be taken as $\gamma = q/T$ and out of all distributions in the class (2.13), the "normal" one is the exponential distribution. For it $P(0) = 0$, $P(x) = 1$ for $x > 0$, $\omega(z) = z$. All other distributions arise as a result of "pathological" variation of $F(x)$ as we take the limit. For convergence to an exponential distribution (in the case of finite average $E[\zeta] = T$), it is necessary and sufficient that for every $x > 0$

$$\lim \int_{x}^{\infty} \frac{t}{q} \, dF \left(\frac{tT}{q} \right) = 0 \qquad \#$$

The following theorem replaces the condition $q_0 \to 0$, which is difficult to verify, by a more practical one.

Theorem 2 (Solovyev, 1971)

If for an arbitrary $p \in (1,2]$, $E[\zeta^p] < \infty$,

$$\lim_{\alpha_p \to 0} P\{\frac{q\tau}{T} > x\} = \lim_{\bar{\alpha}_p \to 0} P\{\frac{q\tau}{T} > x\} = e^{-x} \qquad \# \qquad (2.14)$$

In applications often the regeneration period ζ^* has the following structure: $\zeta^* = \zeta' + \zeta''$, where ζ' is a random period corresponding to system operation in the absence of element failures(so-called "free" period); ζ'' starts with a failure of some system element and ends either by a return to a "brand new" state of the system or by a SF (ζ'' is called "busy" period). In all subsequent examples ζ'' will correspond to system

operation when some of its elements are being repaired. The period ζ'', on the average, is very small in comparison with $E[\zeta']$, which reflects a typical "fast repair" situation.

The following theorem given in   Gnedenko&Solovyev,1974,provides simple sufficient conditions for the asymptotic exponentiality.

Theorem 3.

If $\zeta' \sim Exp(\lambda_0)$, then

$$\lim_{\lambda_0 \cdot E[\zeta''] \to 0} \lambda_0 q\tau > x\} = e^{-x} \qquad \# \qquad (2.15)$$

The most difficult part of applying the theorems given above to particular situations is (1) checking the conditions providing asymptotic exponentiality and (2) finding the normalizing factor for the r.v. τ. The latter demands, as a rule, considerable analytic efforts and involves technical "tricks".

2.4. Applications of theorems of 2.3 to reliability problems.

We survey in this section several important examples of applying the general theorems given in 2.3 to particular reliability models.

Example 1. A $GI|G|r|(m-r)$-system (Solovyev, 1970).

A system has $(m+1)$ identical elements. One and only one of them is operating and all others are in a "cold" standby. The lifetime of the operating element is $X \sim F(x)$, $E[X] = 1$. When this element fails,

it enters the repairing device which has r identical repair channels, each one being able to repair one failed element. The place of the failed element is taken by an other element from the standby. The repair time for each element is $Y \sim G(x)$. If no repair channel is available for a failed element, it will wait in a line. Thus, the normal circulation of elements is operation-waiting for repair-repair-standby-operation, etc.

Assume that at the initial moment $t = 0$ the functioning element has failed and all other m elements were in standby. SF happens when at the instant of the failure of the operating element, all other elements are either in repair or waiting for it, i.e., the standby is empty. Speaking in terms of an $GI|G|r|(m-r)$-system, the SF appears when a customer arrives to a service device when all r service positions and all $m-r$ waiting places are occupied. Let $\kappa(t)$ be the number of customers in service at time t, i.e. the number of failed elements at time t.

Time instants t_i, $t_0 = 0 < t_1 < t_2 < \ldots$ at which elements fail in the presence of r empty service channels are the regeneration points of $\kappa(t)$.

Assume that element failures took place at the instants $\tau_1 = 0$, $\tau_2 = y_1$, $\tau_3 = y_1 + y_2, \ldots, \tau_{m+1} = y_1 + \ldots + y_m$. Consider the following trajectory leading to a SF: i^{th} service channel is busy during the time interval $[\tau_i, \tau_{m+1}]$, $i = 1, \ldots, r$. It is clear that at the instant τ_{m+1} the system will fail. The event $M = \{$no repair has been completed before the $(m+1)^{th}$ failure$\}$ has the probability

$$P\{M\} = q_0 = \int_0^\infty \ldots \int_0^\infty \bar{G}(y_1 + \ldots + y_m) \ldots \bar{G}(y_r + \ldots + y_m) dF(y_1) \ldots dF(y_m) . \qquad (2.16)$$

The following proposition which can be found in Solovyev's paper, 1971, gives sufficient conditions for asymptotic exponentiality of r.v. τ, the time to SF, and the form of the normalizing factor.

If

    (i) $E[X] = 1$, $F(x)$ is fixed, $|F'(x)| \leq C$, $F'(0) = \lambda$,

        $F'(x)$ is continuous at zero;

    (ii) $G(x) = G^0(\frac{x}{\epsilon})$, $G^0(x)$ is fixed and $\epsilon \to 0$;

    (iii) $\alpha_{m+1} = \int_0^\infty x^{m+1} dG^0(x) < \infty$,

Then

$$\lim_{\epsilon \to 0} P\{q\tau > x\} = e^{-x} \quad , \text{ where}$$

$$q \sim q_0 \sim (\lambda\epsilon)^m \int_0^\infty \frac{x^{m-r}}{(m-r)!} \left[\int_x^\infty \bar{G}^0(u)du\right]^{r-1} \frac{G^0(x)}{(r-1)!} dx . \qquad \#$$

(ii) expresses the fact that the service is "fast"; (iii) is used in the proof that $q \sim q_0$.

Example 2 - a general model of standby with renewal (Gnedenko & Solovyev, 1974).
Let us consider the system of Example 1 with the following modification: each element which is not in the repair and is not waiting for repair can fail and has failure rate λ_k depending on the total number of nonfailed

elements k. Regeneration points of $\kappa(t)$ are the instants when $\kappa(t)$

enters state 0. The regeneration period is $\zeta^* = \zeta' + \zeta''$, where ζ', ζ''

are independent, ζ' is a "free" period for which $\kappa(t) = 0$ and ζ''

is a "busy" period for which $\kappa(t) > 0$. Clearly, $\zeta' \sim \text{Exp} (\lambda_0)$. Denote

by q the probability that the SF appears on a busy period; let

$T = \int\limits_0^\infty x \, dG(x)$. The following theorem is the key for investigating this

example.

Theorem (Solovyev & Gnedenko, 1974).

$$\lim_{T \to 0} P\{\lambda_0 q\tau > x\} = e^{-x} . \qquad (2.17)$$

Proof. Let us consider a "majorant" process $\hat{\kappa}(t)$ with respect to $\kappa(t)$,

which is constructed as follows. Replace all element failure rates by

$\bar{\lambda} = \max\limits_k \lambda_k$, and replace r service channels by one channel. The busy

period $\hat{\zeta}''$ for $\hat{\kappa}(t)$ will exceed, on the average, the budy period ζ'':

$E[\hat{\zeta}''] \geq E[\zeta'']$. From the queueing theory it is known that

$E[\hat{\zeta}''] = T/(1-\bar{\lambda}T)$. Since $E[\hat{\zeta}''] \to 0$ as $T \to 0$, $E[\zeta''] \to 0$, and it remains

to apply theorem 3 of Section 2.3.   #

It should be noted that introducing a "majorant" random process is a

very typical way of proving theorems similar to the above. The following

condition reflects "fast" repair:

$$\int\limits_0^\infty y^{m+1} \, dG(y)/T^m \to 0 . \qquad (2.17a)$$

Obviously, it guarantees that $T \to 0$. Similar to the case of Example 1, it was proved that given (2.17a), $q \sim q_0 = P\{M\} = P$ {no repair has been completed on the busy period}.

Simple formulas were derived for q_0 for particular cases $r = m$ and $r = 1$:

For $r = m$

$$q_0 \sim \frac{\lambda_1 \lambda_2 \ldots \lambda_m}{m!} \left(\int_0^\infty x dG(x) \right)^m , \qquad (2.18)$$

For $r = 1$

$$q_0 \sim \frac{\lambda_1 \lambda_2 \ldots \lambda_m}{m!} \int_0^\infty x^m dG(x) . \qquad (2.19)$$

Example 3 - "hot" standby with renewal (Solovyev, 1971). The system consists of m elements each of which operates, fails, is repaired, operates again, etc., independently of all other elements. The state of the system at instant t is described by a vector $\underline{x}(t) = (x_1(t), \ldots, x_m(t))$, where $x_i(t) = 1$ if element i is in operating condition at time t and $x_i(t) = 0$, otherwise. Let us suppose that the set E of 2^m states of the system is partitioned into two subsets $E = E_+ + E_-$, where E_+ is the set of "good" states and E_- is the failure set. The problem is to find an asymptotic distribution of τ, the system failure-free operation time: $\tau = \inf\{t:\underline{x}(t) \in E_- |\underline{x}(0) = (1,1,\ldots,1)\}$.

Assume that the lifetime of ith element $X_i \sim Exp(\lambda_i)$ and that the repair time $Y_i \sim G_i(y)$. For simplicity, $\sum_1^m \lambda_i = 1$. The following notation will be used:

$$T_i = E[Y_i], \quad G(t) = \sum_1^m \lambda_i G_i(t), \quad T_0 = \sum_1^m \lambda_i T_i, \quad m_i = E[Y_i^2], \quad m_0 = \sum_1^m \lambda_i m_i .$$

The regeneration cycle of $\underline{x}(t)$ consists of two independent parts: ζ', on which $\underline{x}(t) = (1,1,\ldots,1)$ and ζ'', on which $\underline{x}(t) < (1,1,\ldots,1)$.

The fast repair situation is reflected by the demand that $m_0 \to 0$. It can be proved that in this case $\bar{\alpha}_2 \to 0$ and $E[\zeta''] \to 0$. The main tool for proving these facts is again replacing the original $\underline{x}(t)$ by another process $\hat{\underline{x}}(t)$ which in this case corresponds to a number of customers in an $M|M|1$-system with input flow with parameter $\lambda = 1$. Note that $E[\zeta' + \zeta''] = 1 + E[\zeta''] \to 1$ as $m_0 \to 0$. Thus, theorem 2 of 2.3 shows that

$$\lim_{m_0 \to 0} P\{q\tau > x\} = \bar{e}^x . \tag{2.20}$$

It is more difficult to find an asymptotic estimate of q. It turns out that if $\sum_{k=1}^m \left(\frac{m_k}{T_k}\right) \to 0$, then

$$q \sim q' = \sum_{\underline{x} \in E_+} \lambda(\underline{x}) \prod_{i=1}^n (\lambda_i T_i)^{1-x_i} , \tag{2.21}$$

where $\lambda(\underline{x})$ is system failure rate for the state $\underline{x} \in E_+$.

Brown, 1975, considers a special case of the system described in
Example 3, for which E_- has *only one state*, namely $\underline{x} = (0,0,\ldots,0)$
and all repair times $Y_i \sim \exp(\mu_i)$. The author proves, using mainly
the Laplace transform technique, the *following theorem*.

Theorem (Brown, 1975)

If all $\lambda_i > 0$, at least one $\lambda_i < \infty$, all $\mu_i < \infty$, at least one
$\mu_i > 0$, and λ_i, μ_i, m change in such a way that

$$\frac{\log m \cdot \sum_{1}^{m} \mu_i}{\prod_{1}^{m} \frac{\lambda_i + \mu_i}{\lambda_i} \cdot \{ \min_{1 \le i \le m} (\lambda_i + \mu_i)\}} \to 0 , \tag{*}$$

then $\tau/E[\tau]$ converges in distribution to an exponential r.v. with
parameter 1   #

Brown's condition (*) reflects the "fast" repair.

Example 4 - renewable system with redundancy (Gnedenko & Solovyev, 1975)

Let us consider the system of Example 3 with some additional features
added. If the state of the system is $\underline{x}(t)$ at time t, then the
probability of the failure of the ith element in the interval $[t,t+h]$ is
$\lambda_i[\underline{x}(t)]h + o(h)$, i.e., this probability depends on the state of the whole
system $\underline{x}(t)$. Besides, there are r, r $\le$ m, repair channels, the repair
time of element i on the repair channel j is $Y_{ij} \sim G_{ij}(y)$, and
$E[Y_{ij}] = T_{ij}$. It is assumed that for each element i there is a set

of repair channels able to perform its repair. The repair of failed elements begins either at the instant of its failure (if the appropriate channel is free) or at the instant at which a repair of some other element has been completed.

It will be assumed that element failure rates are small, formally that they have the form $\alpha\lambda_i(\underline{x})$, where $\lambda_i(\underline{x})$ are fixed and $\alpha \to 0$. Let $\lambda(\underline{x}) = \sum_{i=1}^{m} \lambda_i(\underline{x})$, where $\lambda_i(\underline{x}) = 0$ if $x_i = 0$.

As in the previous cases, the problem is to find the asymptotic distribution of r.v. τ

$$\tau = \inf\{t:\underline{x}(t) \in E_-|\underline{x}(0) = (1,1,\ldots,1)\}$$

The *following theorem based on theorem 3 of 2.3 establishes the asymptotic exponentiality.*

Theorem (Gnedenko & Solovyev, 1975)

Let $\bar{\lambda} = \max_{\underline{x}\in E_+} \lambda(\underline{x})$, $G(y) = \min_{i,j} G_{ij}(y)$, $T = \int_0^\infty \bar{G}(y)dy$.

Then

$$\lim_{\bar{\lambda}T\to 0} P\{\frac{\tau}{E[\tau]} > x\} = \bar{e}^{x} \qquad \# \qquad (2.22)$$

It is intuitively clear that $E[\tau] = (\lambda(\underline{x}(0))q)^{-1}$, where q is the SF probability on a single regeneration period. It will be very instructive to follow the main lines of finding an asymptotic expression for q.

Let us consider the class D of all trajectories of $\underline{x}(t)$ leading from the state $\underline{x} = (1,1,\ldots,1)$ into the failure set E_-. A trajectory $d = \{\underline{x}(0) \to \underline{x}^{(1)} \to \ldots \to \underline{x}^{(k)}\}$ is called monotone if $\underline{x}^{(0)} > \underline{x}^{(1)} > \ldots \ldots > \underline{x}^{(k)}$. For example, the trajectory $(1,1,1) \to (1,1,0) \to (0,1,0)$ is monotone. Denote $||\underline{x}|| = m - \sum_{i=1}^{m} x_i$ ($||\underline{x}||$ is the number of failed elements corresponding to the state $\underline{x}$).

The state $\underline{x}' \in E_-$ for which $||\underline{x}'|| = s = \min_{\underline{x} \in E_-} ||\underline{x}||$ is called minimal system failure state, or minimal state. A monotone path leading from $x = (1,1,\ldots,1)$ to a minimal state is called a minimal monotone path. Let D_0 be the set of all such paths. Then

$$D = D_0 + D_1 , \qquad (2.23)$$

where D_1 is the class of all other paths. Let $q(d)$ be the probability that the SF will occur along a path d. Then

$$q = \sum_{d \in D} q(d) = \sum_{d \in D_0} q(d) + \sum_{d \in D_1} q(d) . \qquad (2.24)$$

The crucial fact is that for $\alpha \to 0$, $q \sim q_0 = \sum_{d \in D_0} q(d)$. In other words, the "main" part of q corresponds to the "shortest" way of failure appearance, i.e., along the minimal monotone paths.

Gnedenko & Solovyev, 1975, present several explicit formulas for q_0 for important particular cases. For example, if $G_{ij}(y) = G_i(y)$,

$E[y_{ij}] = T_i$, $\lambda_i(\underline{x}) = \lambda_i$, each element can be repaired on any channel and $r = m$, then

$$(E[\tau])^{-1} \sim \alpha^s \sum \lambda_{k(1)} \lambda_{k(2)} \cdots \lambda_{k(s)} T_{k(1)} \cdots T_{k(s)} (T_{k(1)}^{-1} + \ldots + T_{k(s)}^{-1}) , \qquad (2.25)$$

where the sum is taken over all minimal system failure·states, and $k(1) < k(2) < \ldots < k(s)$ are the numbers of failed elements in the minimal failure state.

If $r = 1$, then

$$(E[\tau])^{-1} \sim \alpha^s \sum \lambda_{k(1)} \lambda_{k(2)} \cdots \lambda_{k(s)} (m_{s-1,k(1)} + \ldots + m_{s-1,k(s)}), \qquad (2.26)$$

where $m_{s-1,i} = E[Y_i^{s-1}]$ and the sum is similar to that in (2.25).

A problem which is very similar to that described in Example 4 was considered in a less general setting by *Ovchinnikov, 1976.*

A useful summary of main reliability indices of various types of systems, for which the asymptotic exponentiality of time to failure is valid, can be found in the book by Kozlov & Ushakov, 1975, Sec. 4.2.

2.5. <u>Reliability models in terms of queueing theory</u>. The models considered in Section 2.4 can be translated into the language of queueing theory. When the system is normally functioning it generates a flow of failed elements ("customers") which need to be served (repaired). SF appears when an element fails in the presence of, say, m elements which have already failed. In queueing theory terms, this corresponds to the loss of a customer who had arrived to a servicing device when all r service

channels and all (m-r) waiting places are occupied.

The difference between reliability and queueing problems is in the fact that reliability theory is usually interested in the case of the low intensity traffic (this corresponds to fast repair), while the classical queueing asymptotics deals with high traffic intensity.

The paper of Vinogradov, 1967, deals with a $GI|M|1|(n-1)$-system. The interarrival time has d.f. $F(y)$ which varies that

$$\alpha_0 = \int_0^\infty e^{-t} dF(t) \to 0.$$

Let τ_n be the moment of the first loss of a customer. The general class of limiting distributions for the r.v. $\tau_n \cdot \gamma(\alpha_0)$ is investigated. In particular, it is proved that if

$$\int_0^\infty t^2 dF(t) < \infty \quad \text{and} \quad \alpha_0 \to 0 \quad \text{then} \quad P\{\tau_n/E[\tau_n] > x\} \to e^{-x}. \text{ The paper}$$

by Vinogradov, 1974, studies an $M|G|1$-system with fast service: the service distribution time approaches zero in probability. It is proved that $P\{\tau_n/E[\tau_n] > t\} \to e^{-t}$, where τ_n is the instant at which the queue length reaches the level n for the first time.

Solovyev & Zaitsev, 1975, consider an $M|G|1|(k-1)$-system with a Poisson input flow having a variable parameter $h^*(t)$. The service time $Y \sim G^*(y)$. The problem of finding the asymptotic distribution of r.v. τ_k, the time of the first loss of a customer, is investigated. The central result obtained is the following

<u>Theorem</u>. If $h^*(t) = h(t)/\beta$, $G^*(t) = G(t/\alpha)$, $\gamma = \alpha/\beta$, $\beta = \gamma^k$,

$$m_k = \int_0^\infty t^k dG(t), \text{ then}$$

$$\lim_{\gamma \to 0} P\{\tau_k > t\} = \exp\{-\frac{m_k}{k!} \int_0^t h^{k+1}(x)dx\} . \qquad \#$$

Asymptotic analysis for a nonstationary input flow involves considerable analytical difficulties. They are caused mainly by a formal necessity to avoid   pathological behavior of $h^*(t)$.

An investigation of an $M|G|r$-system with a limited waiting space and a nonstationary input flow was carried out by Zaitsev & Solovyev, 1975.

2.6. <u>Exponentiality in Birth-and-Death Processes</u>. Birth-and-Death (BD) process is perhaps the most useful type of random processes in applied probability in general and in reliability theory in particular. The central role in applications is played by the r.v. τ_{0m}, the passage time from state 0 (a "new" system) to state m representing the failure of the system. We survey in this section several basic results concerning the asymptotic behavior of the r.v. τ_{0m} and related r.v.'s.

Let us introduce some notation for a BD-process. The process itself is denoted by $\nu(t)$, its state space $X = (0,1,2,\ldots,m,m+1,\ldots)$. λ_n, $n \geq 0$, μ_n, $n \geq 1$ denote the "upward" and "downward" transition rates. Quantities θ_i, $i \geq 0$, are defined as

$$\theta_0 = 1, \quad \theta_n = \lambda_0 \lambda_1 \cdots \lambda_{n-1}/\mu_1 \mu_2 \cdots \mu_n, \, n \geq 1 ,$$

and

$$\tau_{km} = \inf\{t:\nu(t)=m|\nu(0)=k\} \ .$$

Expectations of τ_{km} are $E[\tau_{km}] = \bar{\tau}_{km}$. It will be assumed that the
BD-process is ergodic.

Let us consider an example from reliability illustrating the use
of BD-processes.

Example - a system with standby and renewal (Gnedenko et al., 1969).

A system has $N = n+d+\ell+s$ similar elements. n elements must always
work and their failure rate is λ. d elements are in a "hot"
(preoperation) standby and they have the same failure rate. ℓ elements
are in a "warm" standby and have failure rate β, $\beta < \lambda$. s elements are
in storage (cold standby) and their failure rate is 0. Failed elements
enter a repair shop which is able to repair simultaneously not more than
r elements. The repair time for each element has an exponential
distribution with parameter μ. If repair facilities are busy, failed
elements wait in a line. Each operating element, which has failed,
should be immediately replaced by an element from the preoperating

standby; its place, in turn, is taken by an element from the warm
standby. Elements in warm standby are replaced by elements from
storage. Elements leaving the repair shop join the storage.

The state of the system at time t can be described by the number
$\nu(t)$ of failed elements. $\nu(t)$ is a BD-process with transition rates
(λ_k, μ_k):

$$\lambda_k = \begin{cases} (n+d) \cdot \lambda + \beta\ell, & 0 \leq k < s \; ; \\ (n+d) \cdot \lambda + \beta(\ell+s-k), & s < k \leq s + \ell \; ; \\ (n+d + \ell + s - k)\lambda, & s + \ell < k \leq N \; . \end{cases}$$

(2.26)

$$\mu_k = \begin{cases} k\mu, & k \leq r \\ r\mu, & r > r \; . \end{cases}$$

System failure in this case is the event that $\nu(t)$ reaches the state
$m = d + \ell + s + 1$, which means that the number of operating elements
dropped below n. So, the main reliability index is expressed through
d.f. of r.v. τ_{0m}.

It is not difficult to find the exact distribution of τ_{0m} by
using ordinary methods, but all computations are very cumbersome.
Usually, the parameters λ_i, μ_i and the level m are such that
reaching m by $\nu(t)$ is a "rare" event. Thus, it is natural to
investigate the asymptotic behavior of τ_{0m}.

For a simple case of fixed λ_n, μ_n and $m \to \infty$, the limiting
distribution of τ_{0m} can be obtained as a corollary of Keilson's

theorem (see Section 2.2). Indeed, let $X_1 = (0,1,2,\ldots,m-1)$, $X_2 = (m,m+1,\ldots)$, $x_0 = 0$; clearly, $q_m \to 0$ for an ergodic process as $m \to \infty$, and r.v. $\tau_{0m}/\bar{\tau}_{0m}$ will converge in distribution to the exponential r.v. But this approach is of limited practical importance in reliability applications for the following reasons. In reliability, the critical level m, which is usually the number of failed elements, is almost never large; on the other hand, the parameters λ_n, μ_n often depend on the number m (see, e.g., the previous example) and a direct application of Keilson's theorem is impossible.

Necessary and sufficient conditions for the asymptotic exponentiality of τ_{0m} were found by Solovyev, 1972. Two of his principal results are presented in the following

Theorem (Solovyev, 1972)

Suppose that λ_i, μ_i and m vary in an arbitrary way. (i) In order that

$$\lim P\{\frac{\tau_{0m}}{\bar{\tau}_{0m}} > x\} = \bar{e}^x \tag{2.27}$$

it is necessary and sufficient that $a_{m,2} \to 0$, where

$$a_{m,2} = \frac{1}{\bar{\tau}_{0m}^2} \sum_{k=0}^{m-1} (\lambda_k \theta_k)^{-1} \sum_{s=0}^{k} \theta_s \bar{\tau}_{0s} . \tag{2.28}$$

(ii)

$$\max_{0 \leq t < \infty} |P\{\tau_{0m} > t\} - \exp(-t/\bar{\tau}_{0m})| \sim a_{m,2} \qquad \# \tag{2.29}$$

The proof of this theorem is based on an ingenious analysis of the distribution of τ_{0m}.

Let us show, following Gnedenko et al., 1969, Section 6.4, how to apply the above theorem. λ_k corresponds to element failure rates and μ_k - to element repair rates. Normally, $\lambda_k \ll \mu_k$ which reflects the "fast" repair case. Formalizing this property, let us write $\lambda_k = \bar{\lambda}_k \alpha$ and assume that $\alpha \to 0$. Obviously,

$$\theta_k = (\bar{\lambda}_0 \ldots \bar{\lambda}_{k-1}/\mu_1 \ldots \mu_k)\alpha^k = \bar{\theta}_k \alpha^k \quad . \tag{2.30}$$

It can be proved that

$$a_{m,2} \sim \bar{\lambda}_{m-1}\bar{\theta}_{m-1}\alpha^m \cdot \sum_1^{m-1} \mu_i^{-1} \quad . \tag{2.31}$$

Thus $\alpha_{m,2} \to 0$ as $\alpha \to 0$ and by the Theorem,

$$P\{\tau_{0m} > t\} \approx \exp\{-t/\bar{\tau}_{0m}\} \quad .$$

The following formula

$$P\{\tau_{0m} > t\} \approx \exp\{-(t-a)/(\bar{\tau}_{0m} - a)\}, \tag{2.32}$$

where $a = \sum_1^{m-1} \mu_i^{-1}$, is more precise and its relative error has the magnitude $(\bar{\tau}_{0m}-t)a/\bar{\tau}_{0m}^2$.

Several important characteristics of a BD-process related to the passage time τ_{0m} such as ergodic and quasi-stationary exit times were studied by Keilson, 1975, 1979.

Ergodic exit time from X_1 to X_2, τ_{Em}, is defined as a passage time from X_1 to X_2, assuming that the initial state distribution for $i \in X_1$ is

$$p_i^E = \pi_i / \sum_{i \in X_1} \pi_i , \qquad i = 0,1,\ldots,m-1 , \qquad (2.33)$$

where π_i is the stationary probability of state i.

Suppose that $\nu(t)$ has been running for a very long time T and all that time it was in the set X_1. Let

$$p_i^Q = \lim_{T \to \infty} P\{\nu(t)=i \,|\, \nu(t') \in X_1, \; t-T \leq t' \leq t\} . \qquad (2.34)$$

It is shown in Keilson, 1979, Sect. 6.6 how the vector $p_-^Q = (p_0^Q,\ldots,p_{m-1}^Q)$ can be expressed through the parameters of BD-process (see Keilson, 1979, Sect. 6.6, 6.5).

Quasi-stationary exit time is defined as a passage time from X_1 to X_2 assuming that the initial state $i \in X_1$ has the probability p_i^Q. This index has a special importance for reliability theory because often it is necessary to predict system reliability given that for a long time it has been working without failures, i.e., formally, it has spent an infinite time in the set X_1.

A surprising fact about τ_{Qm} is that $\tau_{Qm}/\bar{\tau}_{Qm}$ has an <u>exact</u> exponential distribution with $\lambda = 1$. (see Keilson, 1975, 1979). The following theorems summarize the basic facts about the asymptotic behaviour of τ_{0m}, τ_{Em}, τ_{Qm} .

Theorem 1 (Keilson, 1979, Sect. 8.3,8.4)

Let $\nu(t)$ be an ergodic BD-process governed by (λ_n, μ_n) with $X_1 = (0,1,\ldots,m-1)$. Let $\lambda_n/\mu_n \to \rho < 1$ as $n \to \infty$. Then

(i) $\bar{\tau}_{m-1,m}/\bar{\tau}_{0m} \to 1-\rho$ as $m \to \infty$.

(ii) $\bar{\tau}_{0m}$, $\bar{\tau}_{Qm}$, $\bar{\tau}_{Em}$ are asymptotically equal as $m \to \infty$.

(iii) $P\{\tau_{m-1,m}/\bar{\tau}_{0m} > x\} \to (1-\rho)e^{-x}$ as $m \to \infty$ #

Theorem 2 (Keilson, 1979)

For any ergodic BD-process for which $\bar{\tau}_{0m} \to \infty$ as $m \to \infty$, $\tau_{0m}/\bar{\tau}_{0m}$, $\tau_{Em}/\bar{\tau}_{0m}$, $\tau_{Qm}/\bar{\tau}_{0m}$ converge in distribution to r.v. $Y \sim \text{Exp}(1)$.   #

2.7. Main event. Asymptotic invariance.

The most difficult part in finding the normalizing constant for r.v. τ is determining q, the probability of SF on a single regeneration period. Generally, q can be represented in a form of asymptotic series $q = q_0 + q_1 + q_2 + \ldots$, where $q_{k+1} = o(q_k)$. Thus q in the normalizing constant can be replaced by $q_0 \sim q$. The quantity q_0 has a transparent probabilistic meaning and always is a probability of some event termed in the literature as "main" event. Let us return to example 4 in Section 2.4 and, following Gnedenko & Solovyev, 1975, have a closer look at the relation between

$$\sum_{d \in D} q(d) \quad \text{and} \quad \sum_{d \in D_0} q(d).$$

Let $d = \{\underline{x}(0) \to \underline{x}^{(1)} \to \ldots \to \underline{x}^{(s)}\} \in D_0$, and let $k(i)$, $i = 1,2,\ldots,s$, be the numbers of failed elements on the d-trajectory. $q(d)$ can be represented in the following general form:

$$q(d) = \frac{\lambda_{k(0)}(\underline{x}(0))}{\lambda(\underline{x}(0))} \lambda_{k(1)}(\underline{x}^{(1)}) \lambda_{k(2)}(\underline{x}^{(2)}) \ldots \lambda_{k(s-1)}(\underline{x}^{(s-1)}) \alpha^{s-1} \quad \times$$

$$\int \ldots \int_{0 < y_1 < \ldots < y_{s-1}} \bar{G}_{k(0),j(0)}(y_{s-1}) \bar{G}_{k(r_1),j(r_1)}(y_{s-1} - y_1) \ldots \quad \times$$

$$\exp\{-(\lambda(\underline{x}^{(1)})y_1 + \lambda(\underline{x}^{(2)})(y_2 - y_1)\ldots)\alpha\}dy_1 \ldots dy_{s-1} . \tag{2.35}$$

Indeed, the elements with numbers $k(0), k(1), \ldots$ fail at $t = 0$,

$t = y_1, \ldots$; elements with numbers $k(0)$, $k(r_1), \ldots$, begin immediately their repair, and other elements wait for their turn; the repair of those elements which started their repair without waiting does not end before the instant of s-th failure. This corresponds to a SF on the trajectory $d \in D_0$.

From (2.35) it is easy to show that $q(d) \sim \text{Const} \cdot \alpha^{s-1}$, if $\int_0^\infty y^{s-1} dG(y) < \infty$.

Lemma (Gnedenko & Solovyev, 1975)

$$q_1^* = \sum_{d \in D_1} q(d) = o(\alpha^{s-1}) . \tag{2.36}$$

Proof. Obviously, the total number of element failures on any $d \in D_1$ exceeds s. Let $A = \{$more than s element failures preceded the system failure$\}$ and let $B = \{$the regeneration cycle does not end at the instant of $(s+1)^{\text{th}}$ element failure$\}$. Clearly $A \subseteq B$ and $P\{A\} \leq P\{B\}$.

Now replace the random process $\underline{x}(t)$ by a majorant process $\underline{\hat{x}}(t)$ which arises if all failure rates are $\max_{\underline{x}} \lambda(\underline{x}) = \bar{\lambda}$, $G_{ij}(y) = G(y)$ and the number of repair channels $r = 1$. Let $C = \{$the "busy" period did not end at the appearance of $(s+1)^{th}$ failure$\}$. Clearly, $P\{B\} \leqslant P\{C\}$. Let $\eta_0, \eta_1, \ldots, \eta_s$ be the repair times for failures which appear at $0, y_1, \ldots, y_s$ and let $J = \{\eta_0 + \eta_1 + \ldots + \eta_s \geqslant y_s$, $y_s \geqslant 0 \}$. Then $C \subseteq J$, $P\{C\} \leqslant P\{J\}$. Let $G_{s+1}(x)$ denote the d.f. of $\eta_0 + \ldots + \eta_s$. Combining all inequalities we get

$$q_1^* \leqslant P\{J\} = \int_0^\infty \frac{(\bar{\lambda}\alpha)^s x^{s-1}}{(s-1)!} \exp(-\bar{\lambda}\alpha x) d\tilde{G}_{s+1}(x) \leqslant \frac{(\bar{\lambda}\alpha)^s (s+1)^s}{s!} \int_0^\infty y^s dG(y) = O(\alpha^s) ,$$

$$(2.37)$$

which completes the proof. #

The events denoted by M in Examples 1,2 in Section 2.4 are also "main" events.

It is worthwhile to note that the role played by the main event in a renewable system is similar to that played by the event of a failure of all elements constituting minimal cut-sets in a nonrenewable coherent system (see Section 1).

We note that q_0 is the only parameter of the asymptotic distribution which might depend on the properties of repair time d.f. If we check formulas (2.18), (2.25) in Section 2.4 we see that q_0 in both of them depends only on the average value of the repair time. This phenomenon was termed by A.D. Solovyev as "asymptotic invariance" . It takes place only when the main event corresponds to such a system failure history in

which every failed element entered the repair channel without delay (compare, e.g., with (2.19)). It is interesting to note that a similar invariance of stationary probabilities with respect to the form of service time d.f. was established long ago for an important class of queueing systems with losses. Let p_k be the stationary probability that there are k customers in an $M|G|n$-system, $0 \leqslant k \leqslant n$. Sevastyanov, 1957, proved that p_k can be computed by the well-known Erlang formulas which involve only the input flow rate λ and the average service time $\mu = \int_0^\infty x dG(x)$. An up-to-date information about invariance in queueing systems can be found in a recent paper by Dukhovny & Koenigsberg, 1981.

2.8 <u>Bounds on deviation from exponentiality</u>. These bounds are of great interest for engineering applications. Solovyev, 1971, derives an estimate of the rate of convergence to exponential distribution for regenerating process considered in Section 2.3. His main result is the following theorem.

<u>Theorem.</u>

Let $E[\zeta] = 1$, $\phi(t) = P\{q\tau \leqslant t\}$, $2 < p \leqslant 3$, $E[\zeta^k] = m_k$, $E[\zeta^k \chi] = q_k$ then

$$\sup_{y \geqslant 0} |\phi(y) - 1 + e^{-y}| \leqslant C\beta_p/(p-2), \qquad (2.38)$$

where C is an absolute constant and

$$\beta_p = \max[(m_p)^{1/(p-1)}, q_2/m_2, \alpha_p] , \qquad (2.39)$$

α_p is defined in (2.12) # .

For $p = 3$, $C \approx 12$ and in all known cases $\beta_p = O(\alpha_p)$. Denote by $q(x) = P\{A|\zeta = x\}$, the conditional probability of event A given $\zeta = x$. If $q(x) + (x)$, then $\beta_p = \alpha_p$.

Bounds for BD-process can be found in Solovyev, 1972, and one of them is given by (2.29).

Solovyev & Sakhobov, 1976, consider a renewal process of a special form for which the renewal period consists of two independent parts. The first part has an exponential distribution with parameter λ while the second has an arbitrary distribution with finite expectation T. Event A can appear on the second part with probability q, and the appearance of this event is determined completely by the behavior of the process on the second part of the renewal period. Let $W(t) = P\{\tau \leq t\}$, where τ is the moment of the first appearance of event A. It was proved that

$$\exp(-\lambda q t) \leq W(t) \leq \exp(-\lambda q t) + \lambda T . \qquad (2.40)$$

In order to apply this formula it is desirable to have two-sided estimates for the quantity q. Sakhobov & Solovyev, 1977, give such estimates for a renewal process which is related to an $M|G|1$-system for which the input flow intensity depends only on the number $\xi(t)$ of customers in the system at time t. If $\xi(t) = k$, then the probability of an arrival of a new customer in the interval $(t, t+h)$ is

$\lambda_k h + o(h)$. Denote $\tau = \inf\{t : \xi(t) = n+1 \mid \xi(0) = 0\}$. Clearly, $\xi(t)$ is a regenerating process, and the instants of regeneration $t_0, t_1, \ldots,$ are the times when $\xi(t)$ enters state 0. $t_{n+1} - t_n = \zeta' + \zeta''$, ζ' is a free period on which $\xi(t) = 0$ and ζ'' is the busy period where $\xi(t) > 0$. Denote by b_{ij},

$$b_{ij} = \int_0^\infty p_{ij}(t) \lambda_j \bar{G}(t) dt ,$$

where $G(t)$ is the d.f. of service time, $p_{ij}(t)$ is the probability that a pure death process with parameters λ_k passes from state i to state j after time t.

The bounds on q are based on the following lemma which has an independent interest and can also serve to compute q.

Lemma (Sakhobov & Solovyev, 1977).

$$q = c_{1,n} / (c_{1,n} + \ldots, + c_{n,n} + 1) , \qquad (2.41)$$

where $c_{k,n}$ are determined from the following recursive relation:

$$c_{k,n} = b_{k,n} + \sum_{i=k} b_{k,i} c_{i,n} , \quad i \leqslant k \leqslant n , \quad \# \qquad (2.42)$$

Let $\bar{\lambda} = \max_k \lambda_k$, $\lambda_0 = \min_k \lambda_k$, $\gamma = \bar{\lambda} \int_0^\infty \exp(-\lambda_0 t) \bar{G}(t) dt$.

A simple but crude version of a two-sided estimate on q which can be derived from the above lemma is the following inequality: if $2^{n-1}\gamma$ is small, then

$$b_{1,n} \leqslant q \leqslant b_{1,n} / (1 - 2^{n-1} \gamma) . \qquad (2.43)$$

Genis, 1978, investigated the case when the random variable ζ has a Laplace-Stiltjes transform $E[e^{-\zeta z}]$ close to $1/(1+z)$:

$$E[e^{-\zeta z}] = \frac{1+b_1 z+b_2 z^2+b_3 z^3}{1+z+a_2 z^2+a_3 z^3+a_4 z^4} \quad , \quad z = it \ ,$$

where $|a_i|$ and $|b_i|$ are "small".

An application of his method to a "two-period" regenerating process leads to the following estimate:

$$\sup_x |P\{\tau/E[\tau] \leq x\} -1+e^{-x}| < C' \cdot \beta \ ,$$

where

(i) $\beta = \max[\lambda, E[\zeta\chi], \frac{1}{2}\lambda^2 q (E(\zeta^2\chi))^{\frac{1}{2}}, \ \frac{1}{2}\lambda^2 q (E[\zeta^2]),$

$\frac{1}{6}\lambda^3 q^2 \cdot (E[\zeta^3])^{\frac{1}{2}}]$;

(ii) $1/\lambda$ is the average length of the free period; ζ is the length of the busy period; χ is the indicator of SF on the busy period, $q = E[\chi]$, and the free period has an exponential d.f.

(iii) $E[\tau] \sim 1/\lambda q$;

(iv) $0 < C' \leq 4$.

An interesting and quite different approach to measuring the deviation from exponentiality for Markov chains based on the notion of a complete monotone distribution was developed by Keilson, 1979.

Definition. A p.d.f. $f(x)$ on $[0,\infty)$ is completely monotone (write $f \in CM$) if all derivatives of f exist and $(-1)^n f^{(n)}(x) \geqslant 0$, $n \geqslant 1$.

It turns out that $f \in CM \longleftrightarrow f(x) = \int_0^\infty y e^{-yx} \, dG(x)$ for some d.f. $G(x)$, i.e. $f(x)$ is a mixture of exponential densities (see Keilson, 1979, Section 5.3).

The following quantity θ_x serves as a distance to pure exponentiality for r.v. X with p.d.f. $f(x) \in CM$:

$$\theta_x = \frac{\sigma^2}{\mu^2} - 1 , \tag{2.44}$$

where $\mu = E[X]$ and $\sigma^2 = \text{Var } X$.

θ_x is a distance in metric space sense for $f(x) \in CM$ that have finite second moment (Keilson, 1979, Section 8.7). This makes it possible to say for two p.d.f.'s f_1 and $f_2 \in CM$ which of them is "more exponential". The applicability of this distance measure to reliability problems is provided by the fact that many importnat random variables associated with a Markov process have a p.d.f. $\in CM$. Consider a time-reversible ergodic Markoc chain $\nu(t)$ (see Keilson, 1979, Section 2.4, and note that every ergodic BD-process is time-reversible). Decompose the state space of $\nu(t)$, X, into the set X_1 of "good" states and the set X_2 of "bad" states. Consider the passage time τ_{iB} from some $i \in X_1$ to set X_2. It was proved that p.d.f. of τ_{iB} is of CM-type (see Keilson, 1979, Section 8.9). Thus it follows that it is possible to apply the measure of exponentiality (2.44) to r.v.'s

τ_{OB}, τ_{EB}, τ_{QB} – the exit time from the bottom state $0 \in X_1$ to the set X_2, the ergodic exit time from X_1 and the quasi-stationary exit time to X_2 , respectively (see the definitions in Section 2.6).Of course for applications one needs to know the mean value and variance of these r.v.'s. Their computation might be rather complex.Some examples of these computations are given in the above-cited book of J.Keilson.

It would be desirable also to express the deviation from exponentiality of the density $f(x) \in CM$ in an explicit way. The following bound was found by Heyde & Leslie, 1974:

Let $X = Y \cdot W$, $Y \sim Exp(1)$, and W a nonnegative r.v., independent of X with $E[W] = 1$. Then for all $x > 0$

$$\left| P\{X > x\} - e^{-x} \right| \leqslant 8\pi^{-1} \sqrt{3} \, 2^{\frac{1}{2}} (\sigma_X^2 - 1)^{\frac{1}{4}} \tag{2.45}$$

Remark. Assume that

$$\lim_{q \to 0} P\{q\tau/a \leqslant x\} = 1 - e^{-x}, \qquad x > 0,$$

where $a = E[\zeta] = \int_0^\infty (1-F(x))dx$, ζ is a random regeneration period and τ is the time of the first appearance of a "rare" event A.

Note that for any fixed $\varepsilon > 0$ the value of q^* such that $q \leqslant q^* \Rightarrow |P\{q\tau/a \geqslant x\}-e^{-x}| < \varepsilon$ depends, generally, on the form of d.f. $F(x)$. Moreover, it can be proved that for any fixed $q > 0$ and fixed a, no upper bound for $P\{\tau < x\}$ can be found, except for a trivial one. More precisely, for any fixed $a > 0$, $T > 0$, $q > 0$ and $\varepsilon > 0$, one can find a d.f. $H(x)$ such that $\int_0^\infty (1-H(x))dx = a$ but at

the same time, $P\{\tau \leqslant T\} > 1-\varepsilon$. Details on this phenomenon can be found in the paper of Kovalenko, 1973.

3. Kovalenko's Theorem on Asymptotic State Enlargement

A special random process transformation termed "state enlargement" arises in situations when one is dealing with a multidimensional process $\xi(t) = \{\xi_1(t),\xi_2(t)\}$ which has the following property. $\xi_1(t)$ is a slowly varying and $\xi_2(t)$ is a "rapidly" varying component. More specifically, there exists a time interval Δ which is "small" from the viewpoint of $\xi_1(t)$ and "large" from the viewpoint of $\xi_2(t)$. On Δ, the process $\xi_2(t)$ can be investigated for a fixed value of $\xi_1(t)$, say for the value at the initial time in this interval, while on Δ stability is acquired by a certain average characteristic $\bar{\xi}_2(\Delta)$, which in its essential features determines the law of transition from $\xi_1(t)$ to $\xi_1(t+\Delta)$. Following Kovalenko, 1977, 1980, let us consider an important example.

Let $P^{(\alpha)} = ||p_{ij}^{(\alpha)}||$ be the transition matrix of an ergodic Markov chain $\{\nu_n^{(\alpha)}\}$ and let $\{F_{ij}^{(\alpha)}(x)\}$ be a set of distributions of positive r.v.'s. Let $\varepsilon_{ij}^{(\alpha,\beta)}$ be "small" nonnegative numbers such that $\sum\limits_{\beta} \varepsilon_{ij}^{(\alpha,\beta)} \leqslant 1$, and α,β are members of some finite or denumerable set. Let us construct a trajectory of a two-dimensional SMP $\{\alpha(t),\nu(t)\}$ according to the following procedure.

1. Select a pair (α_n, ν_n) as the initial state of the process $\{\alpha(t), \nu(t)\}$ after the nth jump.

2. Select state ν_{n+1} of the second component according to the rule:

$$P\{\nu_{n+1}=j \mid \nu_n\} = p_{\nu_n, j}^{(\alpha_n)}$$

3. Select one-step sojourn time in (α_n, ν_n), $\xi_{\nu_n, \nu_{n+1}}^{(\alpha_n)} \sim F_{\nu_n, \nu_{n+1}}^{(\alpha_n)}(x)$.

4. Using numbers $\epsilon_{\nu_n, \nu_{n+1}}^{(\alpha_n, \beta)}$, simulate the choice of a new state

   α_{n+1}: $\alpha_{n+1} = \alpha_n$ with probability $1 - \sum_{\beta \neq \alpha_n} \epsilon_{\nu_n, \nu_{n+1}}^{(\alpha_n, \beta)}$

   (i.e., α_n remains unchanged); with probability $\epsilon_{\nu_n, \nu_{n+1}}^{(\alpha_n, \beta)}$, $\alpha_{n+1} = \beta$.

5. Remain in state (α_n, ν_n) for the time $\xi_{\nu_n, \nu_{n+1}}^{(\alpha_n)}$ and move afterwards to the state $(\alpha_{n+1}, \nu_{n+1})$.

One can see that $\alpha(t)$ varies "slowly" and $\nu(t)$ varies "rapidly". Moreover, there is a small probability on each step that $\alpha(t)$ will change. Therefore, one could expect that periods of $\alpha(t) = Const$ are approximately exponentially distributed (compare with Keilson's theorem). It turns out that under special circumstances $\alpha(t)$ indeed behaves as a continuous-time Markov chain. More precisely, if a series of conditions related to the properties of all quantities

involved in the description of $\{\alpha(t), \nu(t)\}$ are valid, then the following theorem is true.

Theorem (Kovalenko, 1977)

For any set of disjoint time intervals $\Delta_i = [a_i, b_i]$, $i = 1, \ldots, n$,

$$\sup_{\alpha_1, \ldots, \alpha_n} \left| P\{\alpha(t) = \alpha_i, t \in \Delta_i, i = \overline{1,n}\} - P\{\alpha^*(t) = \alpha_i, t \in \Delta_i, i = 1, n\} \right| \to 0 ,$$

where $\alpha^*(t)$ is a separable Markov process with transition rates

$$\lambda^{\alpha\beta} = \lim \left(\sum_{j,k} \pi_j^{(\alpha)} p_{jk}^{(\alpha)} \varepsilon_{jk}^{(\alpha,\beta)} \Big/ \sum_{j,k} \pi_j^{(\alpha)} p_{jk}^{(\alpha)} \int_0^\infty x dF_{jk}^{(\alpha)}(x) \right) ,$$

where $\pi_j^{(\alpha)}$ are the stationary probabilities for Markov chain $\{\nu_n^{(\alpha)}\}$,

all characteristics of random processes involved depend on ε, and the limit is taken as $\varepsilon \to 0$.   #

Thus, one can see that asymptotically the first component of a two-dimensional random process, $\alpha(t)$, has a simple structure and can be treated separately from $\nu(t)$. One can say that from the viewpoint of the first component, $\alpha(t)$, the states of the second component $\nu(t)$ are indistinguishable, which explains the term "state enlargement".

More formally, let us consider some partition of the state space X of $\underline{x}(t) = \{\alpha(t), \nu(t)\}$ into the sets $\{S_\beta\}$ and a function $g(\underline{x}(t))$ such that $g(\underline{x}) = g(\underline{y})$ if $\underline{x} \in S_\beta$ and $y \in S_\beta$ and $g(\underline{x}) \neq g(\underline{y})$ if $x \in S_{\beta_1}$, $y \in S_{\beta_2}$, $\beta_1 \neq \beta_2$. In the above example, $g(\alpha(t), \nu(t)) = \alpha^*(t)$ and the sets $X_\beta = \{A ; \nu = \beta = Const.\}$, where A is the state space of $\alpha(t)$.

The above-mentioned papers of I.N.Kovalenko contain far reaching generalizations of this scheme and a bibliography of other works related to this topic.

4. "Analyticostatistical" Method of Computing Reliability Characteristics

Let us discuss very briefly an example illustrating a method of computing system reliability characteristics described by Kovalenko, 1976.

Assume, a Markov chain $\{\nu_n, n \geq 0\}$ with state space R is given by an initial distribution and transition probabilities, both of which depend on a small parameter ϵ. Specifically

$$P\{\nu_0 = i\} = p_i^{(0)} = p_i^{(0)}(0) + \epsilon \cdot p_i^{(0)}(1) + \epsilon^2 \cdot p_i^{(0)}(2) + \dots \qquad (4.1)$$

$$P\{\nu_{n+1} = j | \nu_n = i\} = p_{ij} = p_{ij}^{(0)} + \epsilon \cdot p_{ij}(1) + \epsilon^2 \cdot p_{ij}(2) + \dots \qquad (4.2)$$

Let $A \subset R$ be a failure set, and suppose it is important to estimate $E[\zeta]$, the average value of r.v. ζ defined as

$$\zeta = f(\nu_\tau) , \qquad (4.3)$$

where f is some given function and $\tau = \inf\{n : \nu_n \in A\}$.

The special form of Markov chain characteristics (4.1,4.2) allows to obtain a representation of $E[\zeta]$ in series form:

$$E[\zeta] = E[\zeta_0] + \epsilon \cdot E[\zeta_1] + \epsilon^2 \cdot E[\zeta_2] + \dots \qquad (4.4)$$

where ζ_i are random variables. Their distributions depend on some other auxiliary random variables $\{\theta_m\}$ whose d.f.'s involve the parameters of the series (4.1,4.2). Thus, it is possible to simulate r.v.'s $\{\theta_m\}$ and by means of them to obtain relaizations of r.v.'s

$\zeta_0, \zeta_1, \zeta_2, \ldots$. Kovalenko's idea is to estimate $E[\zeta]$ by $\hat{\zeta}$, where

$$\hat{\zeta} = \hat{\zeta}_0 + \varepsilon \cdot \hat{\zeta}_1 + \varepsilon^2 \hat{\zeta}_2 + \ldots \quad , \quad \tag{4.5}$$

and to obtain $\hat{\zeta}_i$, $i \geq 0$, the estimates of ζ_i, using Monte-Carlo simulation. The crucial point is that due to the specific form of (4.1,4.2), only a few terms in (4.5) should be estimated in order to achieve good precision.

Many additional important details of this method can found in the above-cited paper of Kovalenko and also in references mentioned there.

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8